

Correlation additivity relation is superadditive for separable states

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Abstract We deny with a concrete example the generality of the correlation subadditivity relation conjectured by Modi et al's [Phys. Rev. Lett. **104**, 080501 (2010)] for any quantum state and point out that the correlation additivity relation is actually super-additive for separable states. This work indicates that any effort on explicitly proving the conjecture and finding the subadditivity source is unnecessary and fruitless.

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How to characterize and quantify correlations in a quantum system has attracted much attention so far. Very recently K. Modi et al[1] put forward a unified view on quantum and classical correlations. This is a very important contribution to correlation study for it allows to put all correlations on an equal footing and offers a quite concise and clear physics picture (see the figure 1 in Ref.[1] for illustration). Specifically, they defined several derivative states of a given quantum state and made use of their relative entropies to define various correlations. To well demonstrate their unified view, K. Modi et al[1] further showed three examples, where all possible correlations were calculated and the additivity relations were found. The first example is about a comparatively simple family of two-qubit mixed states, i.e., Bell-diagonal states, which comprise of both separable states and entangled states. The second and the third examples aim at the three-qubit W state and the four-qubit cluster state, which represent multipartite entangled pure states. Following the three examples, K. Modi et al[1] wrote: "*From the examples above we conjecture that the correlations of a quantum state are subadditive in the sense $T_\rho \geq E + Q + C_\sigma$. The source of the subadditivity may be due to entanglement being less than the difference in the entropies of ρ and σ , i.e., $S(\sigma) \geq -\text{tr} \rho \log \sigma$. We have not been able to prove this explicitly nor have we found an example showing the contrary.*" **Is the conjecture right?** Specifically, whether the correlation subadditivity relation

$$T_\rho \geq E + Q + C_\sigma \quad (1)$$

holds for any quantum state? If no, then it is surely unnecessary to strictly prove the relation and to find the subadditivity source any longer. In this brief report we will show an example to deny the generality of K. Modi et al's conjecture. As a matter of fact, we find the correlation additivity relation is super-additive for separable states. We think, this work is timely and helpful for those who are still working hard to explicitly prove the subadditivity relation and to find its source.

By the way, for simplicity we will use the same definitions and symbols as those in Ref.[1] hereafter. For all the states in Modi et al's three examples the subadditivity relation $T_\rho \geq E + Q + C_\sigma$ really holds, particularly the first example including some separable states. Based on them, Modi et al then further conjectured that the subadditivity relation holds for any quantum state, too. Is it right? Let us consider the following case. If the quantum state ρ itself is a separable state, then what happens? Since ρ is separable, its closest separable state σ is naturally itself, i.e., $\sigma = \rho$. This means the quantum entanglement E of this state is zero and its quantum mutual information T_ρ becomes T_σ . In this case,

the conjectured subadditivity relation is therefore converted into

$$T_\sigma \geq Q + C_\sigma. \quad (2)$$

In fact, Modi et al[1] have already derived an important equality [i.e., their equation (5)],

$$T_\sigma = Q + C_\sigma - L_\sigma. \quad (3)$$

From equations (2) and (3), one is readily to conclude that, if the conjectured subadditivity relation $T_\sigma \geq Q + C_\sigma$ always holds for any separable state σ , then equation (3) is changed into

$$L_\sigma = 0, \quad T_\sigma = Q + C_\sigma, \quad (4)$$

because all the quantities here including L_σ are relative entropies and should be nonnegative. Note that, in Modi et al's first example, some separable states are definitively included and their L 's are strictly zero indeed. Nonetheless, whether L is constantly zero for any separable state? If yes, then the part about separable states in the figure 1 of Modi et al's paper[1] should be revised at least. Unfortunately, the answer is negative in fact. Let us consider the following simple bipartite quantum separable state

$$\rho = \sigma = \frac{1}{2}(|00\rangle\langle 00| + |1H\rangle\langle 1H|), \quad (5)$$

where $|H\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. After quite complicated and tedious deductions, one can get the closest classical state χ_σ of σ , i.e.,

$$\chi_\sigma = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{4}|10\rangle\langle 10| + \frac{1}{4}|11\rangle\langle 11|. \quad (6)$$

Note that the closest classical state χ_σ is determined completely in virtue of the minimization defined by the equation 3 of Ref.[1], that is, minimizing the quantum dissonance Q . Therefore, Q is simultaneously determined after the minimization, i.e.,

$$Q = \frac{1}{2} = 0.5. \quad (7)$$

Easily one can get the product states

$$\pi_\sigma = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1|), \quad (8)$$

$$\pi_{\chi_\sigma} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|). \quad (9)$$

In terms of Modi et al's definitions, further one can get the quantum mutual information

$$T_\sigma = 1 - \frac{1}{4}[(2 + \sqrt{2})\log_2(2 + \sqrt{2}) + (2 - \sqrt{2})\log_2(2 - \sqrt{2})] \approx 0.601, \quad (10)$$

the classical correlation

$$C_\sigma = \frac{3}{4}\log\frac{4}{3} \approx 0.311, \quad (11)$$

and the quantity

$$L_\sigma = 1 - \frac{3}{4}\log 3 + \frac{1}{4}[(2 + \sqrt{2})\log_2(2 + \sqrt{2}) + (2 - \sqrt{2})\log_2(2 - \sqrt{2})] \approx 0.210. \quad (12)$$

Obviously, $L_\sigma > 0$ and $T_\sigma < Q + C_\sigma$. This indicates that the conjectured subadditivity relation $T_\rho \geq E + Q + C_\sigma$ is not a general conclusion and does not always hold for all quantum states. Since $L_\sigma \geq 0$ for all separable states, the correlation additivity relation is actually superadditive instead of

subadditive for them, specifically, $T_\sigma \leq Q + C_\sigma$. Up to now, naturally it is easy to see that, to strictly prove the correlation subadditivity relation and to find the subadditivity source are unnecessary and fruitless.

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Reference

- [1] K. Modi, T. Paterek, W. Son, V. Vedral, M. Williamson, Phys. Rev. Lett. **104**, 080501 (2010).